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On equivalence picture fuzzy multirelations

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Abstract. A picture fuzzy multirelation over picture fuzzy multiset, defined via (r, s, t)-cut set provides a means of studying the relationships at various thresholds of positive, neutral, and negative membership degrees. In this paper, we have introduced the notion of picture fuzzy equivalence multirelations over picture fuzzy multisets via (r, s, t)-cut set of picture fuzzy multirelations and investigate some of the properties related to them. We have shown that a picture fuzzy multirelation over a picture fuzzy multiset is a picture fuzzy equivalence multirelation if and only if the (r, s, t)-cut set of the picture fuzzy multirelation is an equivalence relation. The picture fuzzy equivalence class with respect to the picture fuzzy equivalence multirelation on a picture fuzzy multiset was defined and it was proved that the two picture fuzzy equivalence classes of a picture fuzzy equivalence multirelation are equal with respect to the (r, s, t)-cut set of picture fuzzy equivalence multirelation. Furthermore, we have established that the intersection of two picture fuzzy equivalence multirelations on a picture fuzzy multiset is again a picture fuzzy equivalence multirelation on the picture fuzzy multiset but their union need not be a picture fuzzy equivalence multirelation. This development of picture fuzzy equivalence multirelation can be employed to form new methodologies for dealing with complex, multi-dimensional relationships in uncertain environments, and also its applications can be explored in decision-making problems, clustering and data analysis.

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1. Introduction

Fuzzy set theory which accommodates only degree of membership was introduced by Zadeh, 1965 [1] as an extension of classical set theory. This theory has numerous extensions and has been applied to various real life problems such as in medicine, economics, engineering etc (See [2, 3, 4, 5, 6] for more details). Zadeh's work was generalised to intuitionistic fuzzy set by Atanassov, 1984 [7] to accommodates the additional degree, degree of nonmembership in addition to the degree of membership of an element of a given set. Many researchers have studied intuitionistic fuzzy set and applied it to real life decision-making problems (See [8, 9, 10] for more details).

In 2013, Cuong and Kreinovich [11] generalised both fuzzy set and intuitionistic fuzzy set to picture fuzzy set. In other theories before the adventure of picture fuzzy set, the degree of neutrality was not incorporated. This important concept can be seen in the voting system where human beings are of the opinions to vote for, vote against, abstain from voting and refusal of voting, and also in medical diagnosis. The theory of picture fuzzy set has been widely studied and applied in various real life problems such as decision-making problem [12, 13, 14, 15, 16], image processing [17] and pattern recognition [18].

Zadeh, [19] generalised the classical relation to fuzzy relation and also introduced the fuzzy versions of reflexivity, symmetric and transitivity which resulted into the equivalence relation. Some of the properties of fuzzy relations and fuzzy equivalence relations were established by Murali and Nemitz in [20] and [21], respectively. In [22], Bustince and Burillo introduced the notion of intuitionistic fuzzy relations and established some of the properties. Some properties of the composition of intuitionistic fuzzy relation were obtained by Deschrijver and Kerre [23]. In [24], Hur et al studied some properties of intuitionistic fuzzy equivalence relations, and also introduced the notion of intuitionistic fuzzy transitive closures and level sets of an intuitionistic fuzzy relation and obtained some of their properties. Rajarajeswari and Una in 2013 [25] introduced the concept of intuitionistic fuzzy multirelations, studied basic properties and the inverse of intuitionistic fuzzy multirelations and obtained reflexitivity, symmetry and transitivity of intuitionistic fuzzy multirelations.

Picture fuzzy relation (PFR) was first introduced by Cuong and Kreinovich [11] as a generalisation of fuzzy relation (FR) and intuitionistic fuzzy relation (IFR). Some properties of composition of PFRs was examined by Phong et al [26] and a new approach for medical diagnosis using composition of fuzzy relations was proposed. Dutta and Saikia [27] introduced equivalence picture fuzzy relation (EPFR) and obtained some of its properties such as equivalence class, intersection and union of equivalence relations via cut set of picture fuzzy sets. Hasan et al [28] defined max-min and min-max compositions for PFRs, some of their properties were investigated and applied in decision making. In [29], Hasan et al defined PFR over PFS, numerous properties related to PFR were established and some operations were discussed with examples. Sangodapo and Nasreen, [30] introduced Picture fuzzy multirelation (PFMR) as an extension of PFR. Some of its basic properties, and inverse of PFMR and their properties were obtained, some operators, Arithmetic mean, Geometric mean and Harmonic mean operators were derived and illustrated with examples to establish both operations and operators. Also, composition of PFMRs was defined and properties associated with them were obtained. Sangodapo [31] contributed to the work of Sangodapo and Nasreen [30] by discussing reflexivity, symmetricity and transitivity of PFMRs over PFMs and some properties associated with them.

In this paper, contribution was made to the work of Sangodapo [31] by introducing picture fuzzy equivalence multirelation (PFEMR) and to obtain its associated properties. Also, Dutta and Saikial's work [27] was generalised from EPFR to PFEMR. Organisation of the paper is as follows; Section 2 reviews the preliminaries on picture fuzzy sets, picture fuzzy multisets and picture fuzzy multirelations. Section 3, introduces the picture fuzzy equivalence multirelation and establishes its associated properties.

2. Preliminaries

In this section, we list some preliminaries of picture fuzzy sets, picture fuzzy multisets and picture fuzzy multirelations.

Definition 2.1 ([11])**.** Let X be a universe. A *picture fuzzy set* (briefly, PFS) N of X is an object of the form

$$N = \{\langle r, \sigma_N(r), \tau_N(r), \eta_N(r)) | r \in X\rangle\},$$

such that $\sigma_N(r) \in [0,1]$ is referred to as the *degree of positive membership*, $\tau_N(r) \in [0,1]$ is called the *degree of neutral membership* and $\eta_N(r) \in [0,1]$ is called the *degree of negative membership* of $r \in N$ and for all $r \in X$,

$$\sigma_N(r) + \tau_N(r) + \eta_N(r) \leq 1$$

and the *degree of refusal membership* of $r \in N$ is $1 - (\sigma_N(r) + \tau_N(r) + \eta_N(r))$.

Definition 2.2 ([11])**.** Let N_1 and N_2 be two PFSs. Then

- $N_1 \subseteq N_2$ if and only if for all $r \in X$, $\sigma_{N_1}(r) \leq \sigma_{N_2}(r)$, $\tau_{N_1}(r) \leq \tau_{N_2}(r)$ and $\eta_{N_1}(r) \geq \eta_{N_2}(r)$,
- $N_1 = N_2$ if and only if $N_1 \subseteq N_2$ and $N_2 \subseteq N_1$,
- $N_1 \cup N_2 = \{(r, \sigma_{N_1}(r) \vee \sigma_{N_2}(r), \tau_{N_1}(r) \wedge \tau_{N_2}(r)), \eta_{N_1}(r) \wedge \eta_{N_2}(r))|\ r \in\ X\}$,
- $N_1 \cap N_2 = \{(r, \sigma_{N_1}(r) \wedge \sigma_{N_2}(r), \tau_{N_1}(r) \wedge \tau_{N_2}(r)), \eta_{N_1}(r) \vee \eta_{N_2}(r))|\ r \in\ X\}$,
- $\overline{N_1} = \{(r, \eta_{N_1}(r), \tau_{N_1}(r), \sigma_{N_1}(r))|\ r \in\ X\}$.

Definition 2.3 ([11])**.** Let N be nonempty set. Then a *picture fuzzy relation* (briefly, PFR) U is a PFS over N defined as

$$U = \{\langle (r_1, r_2), \sigma_N(r_1, r_2), \tau_N(r_1, r_2), \eta_N(r_1, r_2)\rangle | (r_1, r_2) \in N \times N\}$$

with $\sigma_N : N \times N \to [0,1]$, $\tau_N : N \times N \to [0,1]$, $\eta_N : N \times N \to [0,1]$, such that $0 \leq \sigma_N(r_1, r_2) + \tau_N(r_1, r_2) + \eta_N(r_1, r_2) \leq 1$ for every $(r_1, r_2) \in N \times N$.

Definition 2.4 ([32])**.** Let X be a nonempty set. A *picture fuzzy multiset* (briefly, PFMS) N in X is characterised by three functions namely the *positive membership count function pmc*, the *neutral membership count function n_emc* and the negative *membership count function nmc* such that $pmc : X \to W$, $n_emc : X \to W$ and $nmc : X \to W$, respectively, where W is the set of all crisp multisets drawn from $[0,1]$. Thus, for any $r \in X$, pmc is the crisp multiset from $[0,1]$ whose positive membership sequence is defined by $(\sigma_N^1(r), \sigma_N^2(r), \cdots, \sigma_N^n(r))$ such that $\sigma_N^1(r) \geq \sigma_N^2(r) \geq \cdots \geq \sigma_N^n(r)$, $n_e mc$ is the crisp multiset from $[0,1]$ whose neutral membership sequence is defined by $(\tau_N^1(r), \tau_N^2(r), \cdots, \tau_N^n(r))$ and nmc is the crisp multiset from $[0,1]$ whose negative membership sequence is defined by $(\eta_N^1(r), \eta_N^2(r), \cdots, \eta_N^n(r))$, these can be either decreasing or increasing functions satisfying $0 \leq \sigma_N^k(r) + \tau_N^k(r) + \eta_N^k(r) \leq 1 \ \forall \ r \in X, \ \ k = 1, 2, \cdots, n$.

Thus N is represented by

$$N = \{\langle r, \sigma_N^k(r), \tau_N^k(r), \eta_N^k(r)\rangle | r \in X\}$$

$k = 1, 2, \cdots, n$.

Example 2.5. Let $X = \{a, b, c\}$. Then the PFMS N is given as

$$N = \{\langle a, (0.6, 0.3, 0.1), (0.8, 0.05, 0.1)\rangle, \langle b, (0.7, 0.1, 0.2), (0.5, 0.3, 0.2)\rangle, \langle c, (0.4, 0.3, 0.3), (0.65, 0.2, 0.15)\rangle\}.$$

Definition 2.6 ([32])**.** Let

$$N_1 = \{\langle r, \sigma_{N_1}^k(r), \tau_{N_1}^k(r)), \eta_{N_1}^k(r)\rangle | \ r \in \ X\}$$

and

$$N_2 = \{\langle r, \sigma_{N_2}^k(r)), \tau_{N_2}^k(r), \eta_{N_2}^k(r))\rangle | \ r \in \ X\}$$

be two PFMSs drawn from X. Then

- $N_1 \subseteq N_2, \ \Leftrightarrow (\sigma_{N_1}^k(r) \leq \sigma_{N_2}^k(r)), \ (\tau_{N_1}^k(r) \leq \tau_{N_2}^k(r))$ and $(\eta_{N_1}^k(r) \geq \eta_{N_2}^k(r))$, $k = 1, 2, \cdots, n, \ r \in X$,
- $N_1 = N_2, \ \Leftrightarrow N_1 \subseteq N_2$ and $N_2 \subseteq N_1$,
- $N_1 \cup N_2 = \{(r, (\sigma_{N_1}^k(r) \vee \sigma_{N_2}^k(r)), (\tau_{N_1}^k(r) \wedge \tau_{N_2}^k(r)), (\eta_{N_1}^k(r) \wedge \eta_{N_2}^k(r)))| \ r \in \ X\}, \ k = 1, 2, \cdots, n$,
- $N_1 \cap N_2 = \{(r, (\sigma_{N_1}^k(r) \wedge \sigma_{N_2}^k(r))(\tau_{N_1}^k(r) \wedge \tau_{N_2}^k(r)), \ (\eta_{N_1}^k(r) \vee \eta_{N_2}^k(r)))| \ r \in \ X\}, \ k = 1, 2, \cdots, n$,
- $N_1' = \{(r, \eta_{N_1}^k(r), \tau_{N_1}^k(r), \sigma_{N_1}^k(r))| \ r \in \ X\}, \ k = 1, 2, \cdots, n$.

Definition 2.7 ([30])**.** Let N be a nonempty set. Then a *picture fuzzy multirelation* (briefly, PFMR) U on N is PFMS defined by

$$U = \{\langle (r_1, r_2), \sigma_U^k(r_1, r_2), \tau_U^k(r_1, r_2), \eta_U^k(r_1, r_2)\rangle | (r_1, r_2) \in N \times N\}$$

where $k = 1, 2, \cdots, \beta$ (β is the cardinality of the PFMS Z) $\sigma_U^k(r), \tau_U^i(r), \eta_U^k(r)$: $X \to W$, and W is the set of all crisp multisets drawn from $[0,1]$.

Example 2.8. Consider a scenario where a company evaluates his employees proficiency in various skills. Let $N_1 = \{e_1, e_2, e_3\}$ represent employees and $N_2 = \{s_1, s_2, s_3\}$ represents skills. Then the PFMS reflects the proficiency of each employee in each skill, expressed as follows:
$\sigma(n_1, n_2)$ stands for positive membership, representing confidence in the skill,
$\tau(n_1, n_2)$ stands for neutral membership, representing uncertainty and $\eta(n_1, n_2)$ stands for negative membership, representing a lack of proficiency.
Thus PFMR is given as:

$$U = \{\langle ((e_1, s_1), (0.6, 0.2, 0.2)), ((e_1, s_2), 0.8, 0.1, 0.05), ((e_1, s_3), 0.7, 0.1, 0.2), ((e_2, s_1), 0.5, 0.3, 0.2), ((e_2, s_2), 0.65, 0.2, 0.1), ((e_2, s_3), 0.75, 0.05, 0.15), ((e_3, s_1), 0.9, 0.05, 0.04), ((e_3, s_2), 0.85, 0.1, 0.05), ((e_3, s_3), 0.45, 0.4, 0.1)\rangle\}.$$

So for (e_1, s_1),

$\sigma(e_1, s_1) = 0.6$ employee e_1 is highly proficient in the skill s_1, $\tau(e_1, s_1) = 0.2$ employee e_1 is slightly uncertain about $e_1's$ proficient in the skill s_1 and $\eta(e_1, s_1) = 0.2$ employee e_1 lacks proficient in the skill s_1.
For (e_1, s_2),
$\sigma(e_1, s_2) = 0.8$ employee e_1 is highly proficient in the skill s_2, $\tau(e_1, s_2) = 0.1$ employee e_1 is slightly uncertain about $e_1's$ proficient in the skill s_2 and $\eta(e_1, s_2) = 0.05$ employee e_1 has almost no lack of proficiency in the skill s_2.
For (e_1, s_3),
$\sigma(e_1, s_3) = 0.7$ employee e_1 is highly proficient in the skill s_3, $\tau(e_1, s_3) = 0.1$ employee e_1 is slightly uncertain about $e_1's$ proficient in the skill s_3 and $\eta(e_1, s_3) = 0.2$ employee e_1 lacks proficient in the skill s_3, and so on.

Definition 2.9 ([30]). Let $U \in PFMR(N \times N)$. Then the *inverse relation* of U, denoted by U^{-1}, is defined by for all $r_1, r_2 \in N \times N$,

$$\sigma^k_{U^{-1}}(r_2, r_1) = \sigma^k_U(r_1, r_2), \tau^k_{U^{-1}}(r_2, r_1) = \tau^k_U(r_1, r_2), \eta^k_{U^{-1}}(r_2, r_1) = \eta^k_U(r_1, r_2).$$

Definition 2.10 ([30]). Let $U, V \in PFMR(N_1 \times N_2)$. Then $U \subseteq V$ if for every $r_1, r_2 \in N \times N$, $(\sigma^k_U(r_1, r_2) \le \sigma^k_V(r_1, r_2))$, $(\tau^k_U(r_1, r_2) \le \tau^k_V(r_1, r_2))$ and $(\eta^k_U(r_1, r_2) \ge \eta^k_V(r_1, r_2))$; $k = 1, 2, \cdots, n$. If $U \subseteq V$ and $V \subseteq U$, then $U = V$.

Definition 2.11 ([30]). Let $U, V \in PFMR(N_1 \times N_2)$. Then $U \cup V$ is a PFMR such that

$$\sigma^k_{U\cup V}(r_1, r_2) = \bigvee\{\sigma^k_U(r_1, r_2), \sigma^k_V(r_1, r_2)\},$$

$$\tau^k_{U\cup V}(r_1, r_2) = \bigwedge\{\tau^k_G(r_1, r_2), \tau^k_H(r_1, r_2)\}$$

and

$$\eta^k_{U\cup V}(r_1, r_2) = \bigwedge\{\eta^k_U(r_1, r_2), \eta^k_V(r_1, r_2)\}$$

$k = 1, 2, \cdots, n$

Definition 2.12 ([30]). Let $U, V \in PFMR(N_1 \times N_2)$. Then $U \cap V$ is a PFMR such that

$$\sigma^k_{U\cap V}(r_1, r_2) = \bigwedge\{\sigma^k_U(r_1, r_2), \sigma^k_V(r_1, r_2)\},$$

$$\tau^k_{U\cap V}(r_1, r_2) = \bigwedge\{\tau^k_U(r_1, r_2), \tau^k_V(r_1, r_2)\}$$

and

$$\eta^k_{U\cap V}(r_1, r_2) = \bigvee\{\eta^k_U(r_1, r_2), \eta^k_V(r_1, r_2)\}$$

$k = 1, 2, \cdots, n$

Proposition 2.13 ([30]). *Let $U, V, W \in PFMR(Z_1 \times Z_2)$. Then*
(1) $(U^{-1})^{-1} = U$,
(2) $(U \cup V)^{-1} = U^{-1} \cup V^{-1}$,
(3) $(U \cap V)^{-1} = U^{-1} \cap V^{-1}$,
(4) $U \cap (V \cup W) = (U \cap V) \cup (V \cap W)$,
(5) $U \cup (V \cap W) = (U \cup V) \cap (V \cup W)$.

Definition 2.14 ([30]). Let $U = \{\langle(r_1, r_2), \sigma_1^k(r_1, r_2), \tau_1^k(r_1, r_2), \eta_1^k(r_1, r_2)\rangle | (r_1, r_2) \in N \times N\}$ and $V = \{\langle(r_1, r_2), \sigma_2^k(r_1, r_2), \tau_2^k(r_1, r_2), \eta_2^k(r_1, r_2)\rangle | (r_1, r_2) \in N \times N\}$, where $k = 1, 2, \cdots, \beta$ (β is the cardinality of the PFMS N) be two PFMRs on N. Then the *composite relation* $U \circ V$ is a PFMR defined by

$$U \circ V = \{\langle(r_1, r_3), \sigma_1^k \circ \sigma_2^k(r_1, r_3), \tau_1^k \circ \tau_2^k(r_1, r_3), \eta_1^k \circ \eta_2^k(r_1, r_3)\rangle | (r_1, r_3) \in N \times N\},$$

where

$$\sigma_1^k \circ \sigma_2^k(r_1, r_3) = \bigvee\{\sigma_1^k(r_1, r_2) \bigwedge \sigma_2^k(r_2, r_3)\}, \ r_2 \in N,$$

$$\tau_1^k \circ \tau_2^k(r_1, r_3) = \bigwedge\{\tau_1^k(r_1, r_2) \bigwedge \tau_2^k(r_2, r_3)\}, \ r_2 \in N$$

and

$$\eta_1^k \circ \eta_2^k(r_1, r_3) = \bigwedge\{\eta_1^k(r_1, r_2) \bigvee \eta_2^k(r_2, r_3)\}, \ r_2 \in N$$

are called the *positive membership*, *neutral membership* and *negative membership functions*, respectively.

Proposition 2.15 ([30]). *Let $U \in PFMR(N_1 \times N_2)$ and $V \in PFMR(N_2 \times N_3)$. Then $V \circ U$ is in $PFMR(N_1 \times N_3)$.*

Proposition 2.16 ([30]). *Let $U \in PFMR(N_1 \times N_2)$ and $V \in PFMR(N_2 \times N_3)$. Then $(V \circ U)^{-1} = U^{-1} \circ V^{-1}$*

Proposition 2.17 ([30]). *Let $U, V \in PFMR(N_2 \times N_3)$ and $W \in PFMR(N_1 \times N_2)$. Then*
(1) $(V \cap U) \circ W = (V \circ W) \cap (U \circ W)$,
(2) $(V \cup U) \circ W = (V \circ W) \cup (U \circ W)$.

Proposition 2.18 ([30]). *Let $V \in PFMR(N_1 \times N_2)$, $U \in PFMR(N_2 \times N_3)$ and $W \in PFMR(Z_3 \times Z_4)$. Then $(W \circ U) \circ V = W \circ (U \circ V)$.*

Proof.

$$\begin{aligned}
\sigma^k_{(W\circ U)\circ V}(r_1, r_4) &= \bigvee_{r_2}\{\sigma^k_V(r_1, r_2) \wedge (\sigma^k_{W\circ U})(r_2, r_3)\} \\
&= \bigvee_{r_2}\{\sigma^k_V(r_1, r_2) \wedge \{\bigvee_{r_3}\{\sigma^k_U(r_2, r_3) \wedge \sigma^k_W(r_3, r_4)\}\}\} \\
&= \bigvee_{r_2}\{\bigvee_{r_3}\{\sigma^k_V(r_1, r_2) \wedge \{\sigma^k_U(r_2, r_3) \wedge \sigma^k_W(r_3, r_4)\}\}\} \\
&= \bigvee_{r_2}\{\bigvee_{r_3}\{\sigma^k_V(r_1, r_2) \wedge \sigma^k_U(r_2, r_3)\} \wedge \sigma^k_W(r_3, r_4)\} \\
&= \bigvee_{r_3}\{\sigma^k_{U\circ V}(r_1, r_3) \wedge \sigma^k_W(r_3, r_4) \\
&= \sigma^k_{W\circ(U\circ V)}(r_1, r_4)
\end{aligned}$$

$$\begin{aligned}
\tau^k_{(W\circ U)\circ V}(r_1,r_4) &= \bigwedge_{r_2}\{\tau^k_V(r_1,r_2)\wedge(\tau^k_{W\circ U})(r_2,r_3)\}\\
&= \bigwedge_{r_2}\{\tau^k_V(r_1,r_2)\wedge\{\bigwedge_{r_3}\{\tau^k_U(r_2,r_3)\wedge\tau^k_W(r_3,r_4)\}\}\}\\
&= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\tau^k_V(r_1,r_2)\wedge\{\tau^k_U(r_2,r_3)\wedge\tau^k_W(r_3,r_4)\}\}\}\\
&= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\tau^k_V(r_1,r_2)\wedge\tau^k_U(r_2,r_3)\}\wedge\tau^k_W(r_3,r_4)\}\\
&= \bigwedge_{r_3}\{\tau^k_{U\circ V}(r_1,r_3)\wedge\tau^k_W(r_3,r_4)\\
&= \tau^k_{W\circ(U\circ V)}(r_1,r_4)
\end{aligned}$$

$$\begin{aligned}
\eta^k_{(W\circ U)\circ V}(r_1,r_4) &= \bigwedge_{r_2}\{\eta^k_V(r_1,r_2)\wedge(\eta^k_{W\circ U})(r_2,r_3)\}\\
&= \bigwedge_{r_2}\{\eta^k_V(r_1,r_2)\wedge\{\bigwedge_{r_3}\{\eta^k_U(r_2,r_3)\wedge\eta^k_W(r_3,r_4)\}\}\}\\
&= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\eta^k_V(r_1,r_2)\wedge\{\eta^k_U(r_2,r_3)\wedge\eta^k_K(r_3,r_4)\}\}\}\\
&= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\eta^k_V(r_1,r_2)\wedge\eta^k_U(r_2,r_3)\}\wedge\eta^k_W(r_3,r_4)\}\\
&= \bigwedge_{r_3}\{\eta^k_{U\circ V}(r_1,r_3)\wedge\eta^k_W(r_3,r_4)\\
&= \eta^k_{W\circ(U\circ V)}(r_1,r_4)
\end{aligned}$$

□

Definition 2.19 ([31])**.** Let $U=\{\langle(r_1,r_2),\sigma^k_U(r_1,r_2),\tau^k_U(r_1,r_2),\eta^k_U(r_1,r_2)\rangle|(r_1,r_2)\in N\times N\}$, where $k=1,2,\cdots,\beta$ (β is the cardinality of the PFMS N.) Then U is said to be $reflexive$, if

$$\sigma^k_U(r,r)=1,\ \tau^k_U(r,r)=0,\ \text{and}\ \eta^k_U(r,r)=0.$$

$k=1,2,\cdots,\beta$ (β is the cardinality of N) for all $r\in N$.

Proposition 2.20 ([31])**.** *Let $U\in PFMR(N\times N)$ be reflexive. Then*
(1) U^{-1} *is reflexive if and only if* $U=U^{-1}$,
(2) $U\vee V$ *is reflexive for every* $V\in PFMR(N\times N)$,
(3) $U\wedge V$ *is reflexive if and only if* $V\in PFMR(N\times N)$ *is reflexive.*

Proposition 2.21 ([31])**.** *If U and V are reflexive PFMRs, then $U\cup V$ and $U\cap V$ are reflexive PFMRs.*

Definition 2.22 ([31])**.** Let $U\in PFMR(N\times N)$. Then U is said to be $Symmetric$, if

$$\sigma^k_U(r_1,r_2)=\sigma^k_U(r_2,r_1),\ \tau^k_U(r_1,r_2)=\tau^k_U(r_2,r_1),\ \text{and}\ \eta^k_U(r_1,r_2)=\eta^k_U(r_2,r_1).$$

$i = 1, 2, \cdots, \beta$ (β is the cardinality of N) for all $r_1, r_2 \in N$.

Proposition 2.23 ([31]). *If U is symmetric, then U^{-1} is also symmetric.*

Proposition 2.24 ([31]). *Let $U \in PFMR(N \times N)$. Then U is symmetric if and only if $U = U^{-1}$.*

Proposition 2.25 ([31]). *If U and V are symmetric PFMRs, then $U \cap V$ and $U \cup V$ are symmetric PFMRs.*

Proposition 2.26 ([31]). *Given $U \in PFMR(N_1 \times N_2)$ and $V \in PFMR(N_2 \times N_3)$. Then $U \circ V$ is symmetric if and only if $U \circ V = V \circ U$, for symmetric relations U and V.*

Definition 2.27 ([31]). Let $U \in PFMR(N \times N)$. Then U is said to be *transitive*, if $U \circ U \subseteq U$.

Transitivity can also be defined as;

Definition 2.28 ([31]). Let $U \in PFMR(N \times N)$. Then, U is said to be *transitive*, if for every triplet (r_1, r_2, r_3) in $N \times N \times N$ whenever (r_1, r_2) and $(r_2, r_3) \in U$ with certain degrees of relatedness $\sigma^k_U(r_1, r_2)$ and $\sigma^k_U(r_2, r_3)$, $\tau^k_U(r_1, r_2)$ and $\tau^k_U(r_2, r_3)$, $\eta^k_U(r_1, r_2)$ and $\eta^k_U(r_2, r_3)$ then $(r_1, r_3) \in U$ with a degree of relatedness

$$\sigma^k_U(r_1, r_3) \geq \bigwedge\{\sigma^k_U(r_1, r_2), \sigma^k_U(r_2, r_3)\},$$

$$\tau^k_U(r_1, r_3) \leq \bigvee\{\tau^k_U(r_1, r_2), \tau^k_U(r_2, r_3)\}$$

and

$$\eta^k_U(r_1, r_3) \leq \bigvee\{\eta^k_U(r_1, r_2), \eta^k_U(r_2, r_3)\},$$

respectively.

Proposition 2.29 ([31]). *Let U be a transitive relation. Then U^{-1} is transitive if and only if $U = U^{-1}$.*

Proposition 2.30 ([31]). *If U and V are transitive, then $U \cap V$ is transitive but $U \cup V$ not transitive.*

Proof. Suppose U and V are transitive and let $(r_1, r_2, r_3) \in N \times N \times N$. Then we have

$$\sigma^k_{U \cap V}(r_1, r_2) = \bigwedge\{\sigma^k_U(r_1, r_2), \sigma^k_U(r_1, r_2)\},$$

$$\tau^k_{U \cap V}(r_1, r_2) = \bigvee\{\tau^k_U(r_1, r_2), \tau^k_U(r_1, r_2)\}$$

and

$$\eta^k_{U \cap V}(r_1, r_2) = \bigvee\{\eta^k_U(r_1, r_2), \eta^k_U(r_1, r_2)\}.$$

For $\sigma^k_{U \cap V}(r_1, r_3)$,

$$\sigma^k_{U \cap V}(r_1, r_3) = \bigwedge\{\sigma^k_U(r_1, r_2), \sigma^k_U(r_2, r_3)\}.$$

Since U and V are transitive, we have

$$\sigma^k_U(r_1, r_3) \geq \bigwedge\left(\sigma^k_U(r_1, r_2), \sigma^k_U(r_2, r_3)\right).$$

and

$$\sigma_V^k(r_1, r_3) \geq \bigwedge \left(\sigma_V^k(r_1, r_2), \sigma_V^k(r_2, r_3)\right)$$

Thus we get

$$\begin{aligned} &\bigwedge \left(\sigma_U^k(r_1, r_3), \sigma_V^k(r_1, r_3)\right) \\ &\geq \bigwedge \left[\bigwedge \left(\sigma_U^k(r_1, r_2), \sigma_U^k(r_2, r_3)\right), \bigwedge \left(\sigma_V^k(r_1, r_2), \sigma_V^k(r_2, r_3)\right)\right]. \end{aligned}$$

Since

$$\begin{aligned} &\bigwedge \left(\sigma_{U\cap V}^k(r_1, r_2), \sigma_{U\cap V}^k(r_2, r_3)\right) \\ &= \bigwedge \left[\bigwedge \left(\sigma_U^k(r_1, r_2), \sigma_V^k(r_1, r_2)\right), \bigwedge \left(\sigma_U^k(r_2, r_3), \sigma_V^k(r_2, r_3)\right)\right], \end{aligned}$$

we have

$$\sigma_{U\cap V}^k(r_1, r_3) \geq \bigwedge \left(\sigma_{U\cap V}^k(r_1, r_2), \sigma_{U\cap V}^k(r_2, r_3)\right).$$

For $\tau_{U\cap V}^k(r_1, r_3)$,

$$\tau_{U\cap V}^k(r_1, r_3) = \bigvee \{\tau_U^k(r_1, r_2), \tau_U^k(r_2, r_3)\}.$$

Since U and V are transitive, we have

$$\tau_U^k(r_1, r_3) \leq \bigvee \left(\tau_U^k(r_1, r_2), \tau_U^k(r_2, r_3)\right)$$

and

$$\tau_V^k(r_1, r_3) \leq \bigvee \left(\tau_V^k(r_1, r_2), \tau_V^k(r_2, r_3)\right).$$

Thus we get

$$\begin{aligned} &\bigvee \left(\tau_U^k(r_1, r_3), \tau_V^k(r_1, r_3)\right) \\ &\leq \bigvee \left[\bigvee \left(\tau_U^k(r_1, r_2), \tau_U^k(r_2, r_3)\right), \bigvee \left(\tau_V^k(r_1, r_2), \tau_V^k(r_2, r_3)\right)\right]. \end{aligned}$$

Since

$$\begin{aligned} &\bigvee \left(\tau_{U\cap V}^k(r_1, r_2), \tau_{U\cap V}^k(r_2, r_3)\right) \\ &= \bigvee \left[\bigvee \left(\tau_U^k(r_1, r_2), \tau_V^k(r_1, r_2)\right), \bigvee \left(\tau_U^k(r_2, r_3), \tau_V^k(r_2, r_3)\right)\right], \end{aligned}$$

we have

$$\tau_{U\cap V}^k(r_1, r_3) \leq \bigvee \left(\tau_{U\cap V}^k(r_1, r_2), \tau_{U\cap V}^k(r_2, r_3)\right)$$

For $\eta_{U\cap V}^k(r_1, r_3)$,

$$\eta_{U\cap V}^k(r_1, r_3) = \bigvee \{\eta_U^k(r_1, r_2), \eta_U^k(r_2, r_3)\}.$$

Since U and V are transitive, we have

$$\eta_U^k(r_1, r_3) \leq \bigvee \left(\eta_U^k(r_1, r_2), \eta_U^k(r_2, r_3)\right)$$

and

$$\eta_V^k(r_1, r_3) \leq \bigvee \left(\eta_V^k(r_1, r_2), \eta_V^k(r_2, r_3)\right).$$

Thus we get

$$\begin{aligned} &\bigvee \left(\eta_U^k(r_1, r_3), \eta_V^k(r_1, r_3)\right) \\ &\leq \bigvee \left[\bigvee \left(\eta_U^k(r_1, r_2), \eta_U^k(r_2, r_3)\right), \bigvee \left(\eta_V^k(r_1, r_2), \eta_V^k(r_2, r_3)\right)\right]. \end{aligned}$$

Since

$$\begin{aligned} &\bigvee \left(\eta_{U\cap V}^k(r_1, r_2), \eta_{U\cap V}^k(r_2, r_3)\right) \\ &= \bigvee \left[\bigvee \left(\eta_U^k(r_1, r_2), \eta_V^k(r_1, r_2)\right), \bigvee \left(\eta_U^k(r_2, r_3), \eta_V^k(r_2, r_3)\right)\right], \end{aligned}$$

we have

$$\eta_{U\cap V}^k(r_1, r_3) \leq \bigvee \left(\eta_{U\cap V}^k(r_1, r_2), \eta_{U\cap V}^k(r_2, r_3)\right).$$

So $U \cap V$ is transitive.

Next we show that $U \cup V$ is not transitive. This will be done using counter example.

Let $N = \{1, 2, 3\}$. Define relation U as

$$\sigma_U(1,2) = 0.9,\ \tau_U(1,2) = 0.1 \text{ and } \eta_U(1,2) = 0.0,$$
$$\sigma_U(2,3) = 0.8,\ \tau_U(2,3) = 0.1 \text{ and } \eta_U(2,3) = 0.1,$$
$$\sigma_U(1,3) = 0.7,\ \tau_U(1,3) = 0.2 \text{ and } \eta_U(1,3) = 0.1.$$

Also, define relation V as

$$\sigma_V(1,2) = 0.7,\ \tau_V(1,2) = 0.2 \text{ and } \eta_V(1,2) = 0.1,$$
$$\sigma_V(2,3) = 0.6,\ \tau_V(2,3) = 0.3 \text{ and } \eta_V(2,3) = 0.1,$$
$$\sigma_V(1,3) = 0.5,\ \tau_V(1,3) = 0.4 \text{ and } \eta_V(1,3) = 0.1.$$

For U,

$$\sigma_U(1,3) = 0.7 \geq \wedge\{0.9, 0.8\} = 0.8,$$
$$\tau_U(1,3) = 0.2 \leq \vee\{0.1, 0.1\} = 0.1$$

and

$$\eta_U(1,3) = 0.2 \leq \vee\{0.0, 0.1\} = 0.1.$$

For V,

$$\sigma_V(1,3) = 0.5 \geq \wedge\{0.7, 0.6\} = 0.6,$$
$$\tau_V(1,3) = 0.4 \leq \vee\{0.2, 0.3\} = 0.3$$

and

$$\eta_U(1,3) = 0.1 \leq \vee\{0.1, 0.1\} = 0.1.$$

Now, for $U \cup V$,

$$\sigma_{U\cup V}(1,2) = \vee\{0.9, 0.7\} = 0.9,$$
$$\tau_{U\cup V}(1,2) = \wedge\{0.1, 0.2\} = 0.1$$

and

$$\eta_{U\cup V}(1,2) = \wedge\{0.0, 0.1\} = 0.0$$

$$\sigma_{U\cup V}(2,3) = \vee\{0.8, 0.6\} = 0.8,$$
$$\tau_{U\cup V}(2,3) = \wedge\{0.1, 0.3\} = 0.1$$

and

$$\eta_{U\cup V}(2,3) = \wedge\{0.1, 0.1\} = 0.1$$

$$\sigma_{U\cup V}(1,3) = \vee\{0.7, 0.5\} = 0.7,$$
$$\tau_{U\cup V}(1,3) = \wedge\{0.2, 0.4\} = 0.2$$

and

$$\eta_{U\cup V}(1,3) = \wedge\{0.1, 0.1\} = 0.1.$$

So, checking transitivity for $U \cup V$,

$$\sigma_{U\cup V}(1,3) = 0.7 \neq \wedge\{\sigma_{U\cup V}(1,2), \sigma_{U\cup V}(2,3)\} = \wedge\{0.9, 0.8\} = 0.8.$$

Since $0.7 < 0.8$, i.e,

$$\sigma_{U\cup V}(1,3) \neq \wedge\{\sigma_{U\cup V}(1,2), \sigma_{U\cup V}(2,3)\}$$

which implies that $\sigma_{U\cup V}(1,3)$ does not satisfy the transitivity condition. Similarly, $\tau_{U\cup V}(1,3)$ and $\tau_{U\cup V}(1,3)$ satisfy not the transitivity condition. Hence $U \cup V$ not transitive. □

3. Picture fuzzy equivalence multiRelations

Definition 3.1 ([32])**.** Let N be a PFMS of X. Then the (r,s,t)-*cut set* of N, denoted by $\mathcal{C}_{r,s,t}(N)$, is defined as

$$\mathcal{C}_{r,s,t}(N) = \{y \in X|\ \sigma_N^k(y) \geq r,\ \tau_N^k(y) \leq s,\ \eta_N^k(y) \leq t\}.$$

Example 3.2. Let $X = \{a, b, c\}$ and

$$N = \{(a, \{(0.7, 0.2, 0.1), (0.5, 0.3, 0.2)\}), (b, \{(0.6, 0.3, 0.1), (0.4, 0.3, 0.3)\}), (c, \{(0.8, 0.1, 0.1), (0.3, 0.6, 0.1)\})\}.$$

Take $r = 0.6,\ s = 0.4,\ t = 0.2$. Then the (r,s,t)-cut set is

$$\mathcal{C}_{0.6,0.4,.2}(N) = \{(a, (0.7, 0.2, 0.1)), (b, (0.6, 0.3, 0.1)), (c, (0.8, 0.1, 0.1))\}.$$

The (r,s,t)-cut set of N simplifies the multiset by focusing on elements with significant positive, neutral and negative membership degrees.

Definition 3.3 ([27])**.** Let U be a PFR of $X \times X$. Then the (r,s,t)-*cut set* of U, denoted by $\mathcal{C}_{r,s,t}(U)$, is defined as

$$\mathcal{C}_{r,s,t}(U) = \{y_1, y_2 \in X \times X|\ \sigma_U(y_1,y_2) \geq r,\ \tau_U(y_1,y_2) \leq s,\ \eta_U(y_1,y_2) \leq t\}.$$

Definition 3.4 ([30])**.** Let U be a PFMR of $X \times X$. Then the (r,s,t)-*cut set* of U, denoted by $\mathcal{C}_{r,s,t}(U)$, is defined as

$$\mathcal{C}_{r,s,t}(U) = \{(y_1, y_2) \in X \times X|\ \sigma_U^k(y_1,y_2) \geq r,\ \tau_U^k(y_1,y_2) \leq s,\ \eta_U^k(y_1,y_2) \leq t\},$$

where $r, s, t \in [0,1]$ such that $0 \leq r + s + t \leq 1$.

Example 3.5. Let $X_1 = \{a, b\}$, $X_2 = \{c, d\}$ and
$U = \{((a,c), (0.7, 0.2, 0.1)), ((a,d)(0.5, 0.3, 0.2)), ((b,c)(0.8, 0.1, 0.1)), ((b,d)(0.3, 0.6, 0.1))\}$.
Take $r = 0.6,\ s = 0.4,\ t = 0.2$. Then the (r,s,t) cut set is

$$\mathcal{C}_{0.6,0.4,0.2}(U) = \{((a,c)(0.7, 0.2, 0.1)), ((b,c), (0.8, 0.1, 0.1))\}.$$

This (r,s,t)-cut set provides a crisp relation derived from the original PFMR by applying thresholds $r, s,$ and t which can also be used to analyse properties like reflexivity, symmetry or transitivity for further exploration of the multirelation. This can be seen in the next definition with example.

Definition 3.6. A PFMR U is said to be a *picture fuzzy equivalence multirelation* (briefly, PFEMR), if U is reflexive, symmetric and transitive.

Example 3.7. Let $X_1 = \{a, b, c\}$ and

$$U = \{((a,a), (0.9, 0.1, 0.0)), ((b,b)(0.9, 0.05, 0.05)), ((c,c)(1, , 0, 0)), ((a,b)(0.7, 0.2, 0.1)), ((b,a)(0.7, 0.2, 0.1)), ((a,c)(0.8, 0.1, 0.1)),$$

$((c,a)(0.8,0.1,0.1)),((b,c)(0.6,0.3,0.1)),((c,b)(0.6,0.3,0.1))\}$.

Take $r=0.6,\ s=0.4,\ t=0.2$.

Reflexive:

For $(a,a),\ \sigma_U(a,a)=0.9\geq r,\ \tau_U(a,a)=0.1\leq s,\ \eta_U(a,a)=0.0\leq t$.
For $(b,b),\ \sigma_U(b,b)=0.9\geq r,\ \tau_U(b,b)=0.05\leq s,\ \eta_U(b,b)=0.05\leq t$.
For $(c,c),\ \sigma_U(c,c)=1=\geq r,\ \tau_U(c,c)=0\leq s,\ \eta_U(c,c)=0\leq t$.

Then $(a,a),(b,b),(c,c)\in R$. Thus reflexive property holds.

Symmetric:

For $(a,b),\ \sigma_U(a,b)=0.7\geq r,\ \tau_U(a,b)=0.2\leq s,\ \eta_U(a,b)=0.1\leq t$.
For $(b,a),\ \sigma_U(b,a)=0.7\geq r,\ \tau_U(b,a)=0.2\leq s,\ \eta_U(b,a)=0.1\leq t$.
For $(b,c),\ \sigma_U(b,c)=0.6\geq r,\ \tau_U(b,c)=0.3\leq s,\ \eta_U(b,c)=0.1\leq t$.
For $(c,b),\ \sigma_U(c,b)=0.6\geq r,\ \tau_U(c,b)=0.3\leq s,\ \eta_U(c,b)=0.1\leq t$.
For $(a,c),\ \sigma_U(a,c)=0.8\geq r,\ \tau_U(a,c)=0.1\leq s,\ \eta_U(a,c)=0.1\leq t$.
For $(a,c),\ \sigma_U(a,c)=0.8\geq r,\ \tau_U(a,c)=0.1\leq s,\ \eta_U(a,c)=0.1\leq t$.

Then $(a,b),(b,a),(b,c),(c,b),(a,c),(c,a)\in R$. Thus symmetric property holds.

Transitive:

$(a,b),(b,c)\in R\Rightarrow(a,c)\in R$, since

$$\sigma_U(a,c)=0.8\geq r,\ \tau_U(a,c)=0.1\leq s,\ \eta_U(a,c)=0.1\leq t.$$

$(b,c),(c,a)\in R\Rightarrow(b,a)\in R$, since

$$\sigma_U(b,a)=0.7\geq r,\ \tau_U(b,a)=0.2\leq s,\ \eta_U(b,a)=0.1\leq t.$$

$(b,a),(a,c)\in R\Rightarrow(b,c)\in R$, since

$$\sigma_U(b,c)=0.6\geq r,\ \tau_U(b,c)=0.3\leq s,\ \eta_U(b,c)=0.1\leq t.$$

Similarly, we have

$$(a,c),(c,b)\in R\Rightarrow(a,b)\in R,$$
$$(c,a),(a,b)\in R\Rightarrow(c,b)\in R,$$
$$(c,b),(b,a)\in R\Rightarrow(c,a)\in R.$$

Thus transitive property holds. So the (r,s,t)-cut set is

$$\mathcal{C}_{0.6,0.4,0.2}(U)=\{(a,a)(b,b),(c,c),(a,b),(b,a),(b,c),(c,b),(a,c),(c,a)\}.$$

Hence U is a PFEMR.

Theorem 3.8. *Let $U=\{\langle(y_1,y_2),\sigma_U^k(y_1,y_2),\tau_U^k(y_1,y_2),\eta_U^k(y_1,y_2)\rangle|(y_1,y_2)\in X\times X\}$ be a PFMR on X. Then U is a PFEMR on X if and only if $\mathcal{C}_{r,s,t}(U)$ is an equivalence relation on X with $r,s,t\in[0,1]$ and $0\leq r+s+t\leq 1$.*

Proof. Suppose that U is a PFEMR. Then we have

$$\mathcal{C}_{r,s,t}(U)=\{(y_1,y_2)\in X\times X|\ \sigma_U^k(y_1,y_2)\geq r,\ \tau_U^k(y_1,y_2)\leq s,\ \eta_U^k(y_1,y_2)\leq t\}.$$

Thus $\sigma_U^k(y,y)=1\geq r,\ \tau_U^k(y,y)=0\leq s$ and $\eta_U^k(y,y)=0\leq t$ for all $y\in X$. So $(y,y)\in\mathcal{C}_{r,s,t}(U)$ which means that $\mathcal{C}_{r,s,t}(U)$ is reflexive.

Now, let $(y_1,y_2)\in\mathcal{C}_{r,s,t}(U)$. Then $\sigma_U^k(y_1,y_2)\geq r,\ \tau_U^k(y_1,y_2)\leq s$ and $\eta_U^k(y_1,y_2)\leq t$. Since U is a PFEMR, we get

$$\sigma_U^k(y_2,y_1)=\sigma_U^k(y_1,y_2)\geq r,$$
$$\tau_U^k(y_2,y_1)=\tau_U^k(y_1,y_2)\leq s,$$

and

$$\eta_U^k(y_2, y_1) = \eta_U^k(y_1, y_2) \leq t.$$

Thus $(y_2, y_1) \in \mathcal{C}_{r,s,t}(U)$, which means that $\mathcal{C}_{r,s,t}(U)$ is symmetric.

Also, let $(y_1, y_2) \in \mathcal{C}_{r,s,t}(U)$ and $(y_2, y_3) \in \mathcal{C}_{r,s,t}(U)$. Then we have

$$\sigma_U^k(y_1, y_2) \geq r,\ \tau_U^k(y_1, y_2) \leq s \text{ and } \eta_U^k(y_1, y_2) \leq t$$

and

$$\sigma_U^k(y_2, y_3) \geq r,\ \tau_U^k(y_2, y_3) \leq s \text{ and } \eta_U^k(y_2, y_3) \leq t.$$

Thus

$$\sigma_U^k(y_1, y_2) \bigwedge \sigma_U^k(y_2, y_3) \geq r,$$

$$\tau_U^k(y_1, y_2) \bigwedge \tau_U^k(y_2, y_3) \leq s$$

and

$$\eta_U^k(y_1, y_2) \bigvee \eta_U^k(y_2, y_3) \leq t$$

which imply that

$$\bigvee\{\sigma_U^k(y_1, y_2) \bigwedge \sigma_U^k(y_2, y_3)\} \geq r \Rightarrow (\sigma_U^k \circ \sigma_U^k)(y_1, y_3) \geq r$$

$$\bigwedge\{\tau_U^k(y_1, y_2) \bigwedge \tau_U^k(y_2, y_3)\} \leq s \Rightarrow (\tau_U^k \circ \tau_U^k)(y_1, y_3) \leq s$$

$$\bigwedge\{\eta_U^k(y_1, y_2) \bigvee \eta_U^k(y_2, y_3)\} \leq t \Rightarrow (\eta_U^k \circ \eta_U^k)(y_1, y_3) \geq t.$$

Since U is a PFEMR, we get

$$\sigma_U^k(y_1, y_3) \geq (\sigma_U^k \circ \sigma_U^k)(y_1, y_3) \geq r,$$

$$\tau_U^k(y_1, y_3) \leq (\tau_U^k \circ \tau_U^k)(y_1, y_3) \leq s$$

and

$$\eta_U^k(y_1, y_3) \leq (\eta_U^k \circ \eta_U^k)(y_1, y_3) \leq t.$$

So $(y_1, y_3) \in \mathcal{C}_{r,s,t}(U)$, which means that $\mathcal{C}_{r,s,t}(U)$ is transitive. Hence $\mathcal{C}_{r,s,t}(U)$ is an equivalence relation on X.

Conversely, Suppose that $\mathcal{C}_{r,s,t}(U)$ is an equivalence relation on X with $r, s, t \in [0, 1]$ and $0 \leq r + s + t \leq 1$, and let $y \in X$. Then clearly, $(y, y) \in \mathcal{C}_{r,s,t}(U)$. Note that $\mathcal{C}_{1,0,0}(U)$ is an equivalence relation on X. Thus we have

$$\sigma_U^k(y, y) \geq 1,\ \tau_U^k(y, y) \leq 0 \text{ and } \eta_U^k(y, y) \leq 0.$$

So $\sigma_U^k(y, y) = 1,\ \tau_U^k(y, y) = 0$ and $\eta_U^k(y, y) = 0$. Hence U is reflexive.

Next, given $y_1, y_2 \in X$, let $\sigma_U^k(y_1, y_2) = r,\ \tau_U^k(y_1, y_2) = s$ and $\eta_U^k(y_1, y_2) = t$. Then $(y_1, y_2) \in \mathcal{C}_{r,s,t}(U)$. By the hypothesis, $\mathcal{C}_{r,s,t}(U)$ is symmetric. Thus $(y_2, y_1) \in \mathcal{C}_{r,s,t}(U)$. So we have

$$\sigma_U^k(y_2, y_1) \geq r = \sigma_U^k(y_1, y_2),\ \tau_U^k(y_2, y_1) \leq s = \tau_U^k(y_1, y_2) \text{ and } \eta_U^k(y_2, y_1) \leq t = \eta_U^k(y_1, y_2).$$

Similarly, if $\sigma_U^k(y_2, y_1) \geq l,\ \tau_U^k(y_2, y_1) \leq m$ and $\eta_U^k(y_2, y_1) \leq n$, then $(y_1, y_2) \in \mathcal{C}_{l,m,n}(U)$. Now, we have

$$\sigma_U^k(y_1, y_2) \geq l = \sigma_U^k(y_2, y_1),\ \tau_U^k(y_1, y_2) \leq m = \tau_U^k(y_2, y_1) \text{ and } \eta_U^k(y_1, y_2) \leq n = \eta_U^k(y_2, y_1).$$

Hence $\sigma_U^k(y_1, y_2) = \sigma_U^k(y_2, y_1),\ \tau_U^k(y_1, y_2) = \tau_U^k(y_2, y_1)$ and $\eta_U^k(y_1, y_2) = \eta_U^k(y_2, y_1)$. Therefore U is symmetric.

Finally, given $y_1, y_2, y_3 \in X$, let

$$\sigma_U^k(y_1, y_2) \wedge \sigma_U^k(y_2, y_3) = r,$$

$$\tau_U^k(y_1, y_2) \wedge \tau_U^k(y_2, y_3) = s$$

and

$$\eta_U^k(y_1, y_2) \vee \eta_U^k(y_2, y_3) = t,$$

where $r \geq 0,\ s < 1,\ t < 1$ and $0 \leq r + s + t \leq 1$. Then we have

$$\sigma_U^k(y_1, y_2) \geq r,\ \sigma_U^k(y_2, y_3) \geq r,$$

$$\tau_U^k(y_1, y_2) \leq s, \tau_U^k(y_2, y_3) \leq s$$

and

$$\eta_U^k(y_1, y_2) \leq t,\ \eta_U^k(y_2, y_3) \leq t.$$

Thus $(y_1, y_2) \in \mathcal{C}_{r,s,t}(U)$ and $(y_2, y_3) \in \mathcal{C}_{r,s,t}(U)$. Since $\mathcal{C}_{r,s,t}(U)$ is transitive by the hypothesis, $(y_1, y_3) \in \mathcal{C}_{r,s,t}(U)$. So we get

$$\sigma_U^k(y_1, y_3) \geq r,\ \tau_U^k(y_1, y_3) \leq s \text{ and } \eta_U^k(y_1, y_3) \leq t.$$

Furthermore, we have the following implications:

$$\sigma_U^k(y_1, y_3) \geq r = \sigma_U^k(y_1, y_2) \wedge \sigma_U^k(y_2, y_3)$$

$$\tau_U^k(y_1, y_3) \leq s = \tau_U^k(y_1, y_2) \wedge \tau_U^k(y_2, y_3),$$

$$\eta_U^k(y_1, y_3) \leq t = \eta_U^k(y_1, y_2) \vee \eta_U^k(y_2, y_3)$$

$\Rightarrow$

$$\sigma_U^k(y_1, y_3) \geq \vee\{\sigma_U^k(y_1, y_2) \wedge \sigma_U^k(y_2, y_3)\},$$

$$\tau_U^k(y_1, y_3) \leq \wedge\{\tau_U^k(y_1, y_2) \wedge \tau_U^k(y_2, y_3)\}.$$

$$\eta_U^k(y_1, y_3) \leq \wedge\{\eta_U^k(y_1, y_2) \vee \eta_U^k(y_2, y_3)\}$$

$\Rightarrow$

$$\sigma_U^k(y_1, y_3) \geq (\sigma_U^k \circ \sigma_U^k)(y_1, y_3),$$

$$\tau_U^k(y_1, y_3) \leq (\tau_U^k \circ \tau_U^k)(y_1, y_3),$$

$$\eta_U^k(y_1, y_3) \leq (\eta_U^k \circ \eta_U^k)(y_1, y_3).$$

$\Rightarrow \sigma_U^k \supseteq \sigma_U^k \circ \sigma_U^k,\ \tau_U^k \subseteq \tau_U^k \circ \tau_U^k,\ \eta_U^k \subseteq \eta_U^k \circ \eta_U^k$. Hence U is transitive. Therefore U is a PFEMR. □

Definition 3.9. Let $U = \{\langle (y_1, y_2), \sigma_U^k(y_1, y_2), \tau_U^k(y_1, y_2), \eta_U^k(y_1, y_2)\rangle | (y_1, y_2) \in X \times X\}$ be a PFEMR on X and $p \in X$. Then the PFMS pU defined by

$$pU = \{(y, p\sigma_U^k(y), p\tau_U^k(y), p\eta_U^k(y) \mid y \in X)\},$$

where

$$(p\sigma_U^k)(y) = \sigma_U^k p^y,\ p\tau_U^k)(y) = \tau_U^k p^y \text{ and } p\eta_U^k)(y) = \eta_U^k p^y.$$

for all $y \in X$ is called a *picture fuzzy equivalence multiclass* (PFEMC) of p with respect to U.

Example 3.10. Let $X = \{a, b, c\}$ and consider the PFEMR on X given by

$$U = \{((a, a), (0.9, 0.1, 0.0)), ((b, b)(0.9, 0.05, 0.05)), ((c, c)(1.0, , 0.0, 0.0)),$$
$$((a, b)(0.7, 0.2, 0.1)), ((b, a)(0.7, 0.2, 0.1)), ((a, c)(0.8, 0.1, 0.1)),$$
$$((c, a)(0.8, 0.1, 0.1)), ((b, c)(0.6, 0.3, 0.1)), ((c, b)(0.6, 0.3, 0.1))\}.$$

Take $r = 0.6,\ s = 0.3,\ t = 0.1$. Then we have

$$\sigma_U(a, a) \geq 0.6,\ \tau_U(a, a) \leq 0.3,\ \eta_U(a, a) \leq 0.1,$$
$$\sigma_U(a, b) \geq 0.6,\ \tau_U(a, b) \leq 0.3,\ \eta_U(a, b) \leq 0.1,$$
$$\sigma_U(a, c) \geq 0.6,\ \tau_U(a, c) \leq 0.3,\ \eta_U(a, c) \leq 0.1;$$

$\Rightarrow aU = (a, a)(a, b)(a, c) = \{a, b, c\}$

$$\sigma_U(b, b) \geq 0.6,\ \tau_U(b, b) \leq 0.3,\ \eta_U(b, b) \leq 0.1,$$
$$\sigma_U(b, c) \geq 0.6,\ \tau_U(b, c) \leq 0.3,\ \eta_U(b, c) \leq 0.1,$$
$$\sigma_U(b, a) \geq 0.6,\ \tau_U(b, a) \leq 0.3,\ \eta_U(b, a) \leq 0.1;$$

$\Rightarrow bU = (b, b)(b, c)(b, a) = \{a, b, c\}$

$$\sigma_U(c, c) \geq 0.6,\ \tau_U(c, c) \leq 0.3,\ \eta_U(c, c) \leq 0.1,$$
$$\sigma_U(c, a) \geq 0.6,\ \tau_U(c, a) \leq 0.3,\ \eta_U(c, a) \leq 0.1,$$
$$\sigma_U(c, b) \geq 0.6,\ \tau_U(c, b) \leq 0.3,\ \eta_U(c, b) \leq 0.1$$

Thus $aU = bU = cU = \{a, b, c\}$. So U partitions X into equivalence classes, consistent with the properties of reflexivity, symmetry and transitivity.

This ensures that the equivalence class includes element with sufficiently high positive membership and controlled levels of neutral and negative memberships.

Theorem 3.11. *Let* $U = \{\langle (y_1, y_2), \sigma_U^k(y_1, y_2), \tau_U^k(y_1, y_2), \eta_U^k(y_1, y_2)\rangle | (y_1, y_2) \in X \times X\}$ *be a PFEMR on* X *and* $p \in X$. *Then for* $r, s, t \in [0, 1]$ *with* $0 \leq r + s + t \leq 1$, $\mathcal{C}_{r,s,t}(pU) = [p]$ *is the equivalence class of* p *with respect to the equivalence relation* $\mathcal{C}_{r,s,t}(U)$ *on* X.

Proof.

$$\begin{aligned} [p] &= \{y \in X \mid (p^y) \in \mathcal{C}_{r,s,t}(U)\} \\ &= \{y \in X \mid \sigma_U^k p^y \geq r, \tau_U^k p^y \leq s, \eta_U^k p^y \leq t\} \\ &= \{y \in X \mid (p\sigma_U^k)(y) \geq r, (p\tau_U^k)(y) \leq s, (p\eta_U^k)(y) \leq t\} \\ &= \mathcal{C}_{r,s,t}(pU). \end{aligned}$$

□

Theorem 3.12. *Let* $U = \{\langle (y_1, y_2), \sigma_U^k(y_1, y_2), \tau_U^k(y_1, y_2), \eta_U^k(y_1, y_2)\rangle | (y_1, y_2) \in X \times X\}$ *be a PFEMR on* X. *Then* $[p] = [q]$ *if and only if* $(p, q) \in \mathcal{C}_{r,s,t}(U)$ *where* $[p]$ *and* $[q]$ *are equivalence classes of* p *and* q *with respect to the equivalence relation* $\mathcal{C}_{r,s,t}(U)$ *in* X *for* $r, s, t \in [0, 1]$ *with* $0 \leq r + s + t \leq 1$.

Proof. Suppose $[p]=[q]$. Then $\mathcal{C}_{r,s,t}(pU)=\mathcal{C}_{r,s,t}(qU)$. Thus

$$\{y\in X \mid (p\sigma_U^k)(y)\geq r,(p\tau_U^k)(y)\leq s,(p\eta_U^k)(y)\leq t\}$$
$$=\{y\in X \mid (q\sigma_U^k)(y)\geq r,(q\tau_U^k)(y)\leq s,(q\eta_U^k)(y)\leq t\}.$$

Let $y\in\mathcal{C}_{r,s,t}(pU)=\mathcal{C}_{r,s,t}(qU)$. Then we have

$$\begin{aligned}
&(p\sigma_U^k)(y)\geq r,(p\tau_U^k)(y)\leq s,(p\eta_U^k)(y)\leq t \text{ and}\\
&(q\sigma_U^k)(y)\geq r,(q\tau_U^k)(y)\leq s,(q\eta_U^k)(y)\leq t\\
\Rightarrow\ &\sigma_U^k p^y\geq r,\tau_U^k p^y\leq s,\eta_U^k p^y\leq t \text{ and } \sigma_U^k q^y\geq r,\tau_U^k q^y\leq s,\eta_U^k q^y\leq t\\
\Rightarrow\ &(\sigma_U^k p^y\wedge\sigma_U^k q^y)\geq r,(\tau_U^k p^y\wedge\tau_U^k q^y)\leq s,(\eta_U^k p^y\vee\eta_U^k q^y)\leq t\\
\Rightarrow\ &\vee(\sigma_U^k p^y\wedge\sigma_U^k q^y)\geq r,\wedge(\tau_U^k p^y\wedge\tau_U^k q^y)\leq s,\wedge(\eta_U^k p^y\vee\eta_U^k q^y)\leq t\\
\Rightarrow\ &(\sigma_U^k\circ\sigma_U^k)\geq r,(\tau_U^k\circ\tau_U^k)\leq s \text{ and } (\eta_U^k\circ\eta_U^k)\leq t\\
\Rightarrow\ &(p,q)\in\mathcal{C}_{r,s,t}(U).
\end{aligned}$$

Conversely, suppose the necessary condition holds and let $(p,q)\in\mathcal{C}_{r,s,t}(U)$. Then

$$\sigma_U^k(p,q)\geq r,\ \tau_U^k(p,q)\leq s,\ \eta_U^k(p,q)\leq t. \tag{3.1}$$

Let $y\in\mathcal{C}_{r,s,t}(pU)$. Then we have

$$\begin{aligned}
&(p\sigma_U^k)(y)\geq r,(p\tau_U^k)(y)\leq s,(p\eta_U^k)(y)\leq t\\
\Rightarrow\ &\sigma_U^k p^y\geq r,\tau_U^k p^y\leq s,\eta_U^k p^y\leq t\\
\Rightarrow\ &(\sigma_U^k q^p\wedge\sigma_U^k p^y)\geq r,(\tau_U^k q^p\wedge\tau_U^k p^y)\leq s,(\eta_U^k q^p\vee\eta_U^k p^y)\leq t \text{ [By (3.1)]}\\
\Rightarrow\ &\vee(\sigma_U^k q^p\wedge\sigma_U^k p^y)\geq r,\wedge(\tau_U^k q^p\wedge\tau_U^k p^y)\leq s \text{ and } \wedge(\eta_U^k q^p\vee\eta_U^k p^y)\leq t\\
\Rightarrow\ &(\sigma_U^k\circ\sigma_U^k)\geq r,(\tau_U^k\circ\tau_U^k)\leq s \text{ and } (\eta_U^k\circ\eta_U^k)\leq t\\
\Rightarrow\ &(q\sigma_U^k)(y)\geq r,(q\tau_U^k)(y)\leq s \text{ and } (q\eta_U^k)(y)\leq t\\
\Rightarrow\ &(p,q)\in\mathcal{C}_{r,s,t}(qU).
\end{aligned}$$

Thus we get

$$\mathcal{C}_{r,s,t}(pU)\subseteq\mathcal{C}_{r,s,t}(qU). \tag{3.2}$$

Similarly, let $y\in\mathcal{C}_{r,s,t}(qU)$. Then we have

$$\begin{aligned}
&(q\sigma_U^k)(y)\geq r,(q\tau_U^k)(y)\leq s,(q\eta_U^k)(y)\leq t\\
\Rightarrow\ &\sigma_U^k q^y\geq r,\tau_U^k q^y\leq s,\eta_U^k q^y\leq t\\
\Rightarrow\ &(\sigma_U^k p^q\wedge\sigma_U^k q^y)\geq r,(\tau_U^k p^q\wedge\tau_U^k q^y)\leq s,(\eta_U^k p^q\vee\eta_U^k q^y)\leq t \text{ [By (3.1)]}\\
\Rightarrow\ &\vee(\sigma_U^k p^q\wedge\sigma_U^k q^y)\geq r,\wedge(\tau_U^k p^q\wedge\tau_U^k q^y)\leq s \text{ and } \wedge(\eta_U^k p^q\vee\eta_U^k q^y)\leq t\\
\Rightarrow\ &(\sigma_U^k\circ\sigma_U^k)\geq r,(\tau_U^k\circ\tau_U^k)\leq s \text{ and } (\eta_U^k\circ\eta_U^k)\leq t\\
\Rightarrow\ &(p\sigma_U^k)(y)\geq r,(p\tau_U^k)(y)\leq s \text{ and } (p\eta_U^k)(y)\leq t\\
\Rightarrow\ &(q,p)\in\mathcal{C}_{r,s,t}(pU).
\end{aligned}$$

Thus we get

$$\mathcal{C}_{r,s,t}(qU)\subseteq\mathcal{C}_{r,s,t}(pU). \tag{3.3}$$

So by (3.2) and (3.3), $\mathcal{C}_{r,s,t}(qU)=\mathcal{C}_{r,s,t}(pU)$.
Hence $[p]=[q]$. □

Theorem 3.13. *Let*

$$U=\{\langle(y_1,y_2),\sigma_U^k(y_1,y_2),\tau_U^k(y_1,y_2),\eta_U^k(y_1,y_2)\rangle|(y_1,y_2)\in X\times X\}$$

and

$$V=\{\langle(y_1,y_2),\sigma_V^k(y_1,y_2),\tau_V^k(y_1,y_2),\eta_V^k(y_1,y_2)\rangle|(y_1,y_2)\in X\times X\}$$

be two PFEMRs on X. Then $U\cap V$ is a PFEMR on X.

Proof. Let $r, s, tin[0,1]$ with $0 \leq r+s+t \leq 1$. Then we have

$$\mathcal{C}_{r,s,t}(U \cap V) = \mathcal{C}_{r,s,t}(U) \cap \mathcal{C}_{r,s,t}(V).$$

We know that $\mathcal{C}_{r,s,t}(U)$ and $\mathcal{C}_{r,s,t}(V)$ are equivalence relations on X. So $\mathcal{C}_{r,s,t}(U \cap V)$ is also an equivalence relation on X. □

Remark 3.14. Union of two PFEMRs on a set need not be a PFEMR (See Example 3.15).

Example 3.15. Let $N = \{1, 2, 3\}$ and consider two PFEMRs U and V on N defined as follows:

$$\sigma_U(1,1) = \sigma_U(2,2) = \sigma_U(3,3) = 1,$$

$$\tau_U(1,1) = \tau_U(2,2) = \tau_U(3,3) = 0,$$

$$\eta_U(1,1) = \eta_U(2,2) = \eta_U(3,3) = 0,$$

$$\sigma_U(1,2) = \sigma_U(2,1) = 0.8,$$

$$\tau_U(1,2) = \tau_U(2,1) = \tau_U(2,3) = \tau_U(3,2) = \eta_U(1,2) = \eta_U(2,1) = 0.1,$$

$$\sigma_U(1,3) = \sigma_U(3,1) = 0.6,$$

$$\sigma_U(2,3) = \sigma_U(3,2) = 0.7,$$

$$\tau_U(1,3) = \tau_U(3,1) = \eta_U(1,3) = \eta_U(3,1) = \eta_U(2,3) = \eta_U(3,2) = 0.2.$$

Also,

$$\sigma_V(1,1) = \sigma_V(2,2) = \sigma_V(3,3) = 1,$$

$$\tau_V(1,1) = \tau_V(2,2) = \tau_V(3,3) = 0,$$

$$\eta_V(1,1) = \eta_V(2,2) = \eta_V(3,3) = 0,$$

$$\sigma_V(1,2) = \sigma_V(2,1) = 0.8,$$

$$\tau_V(1,2) = \tau_V(2,1) = \eta_V(1,2) = \eta_V(2,1) = 0.05,$$

$$\sigma_V(1,3) = \sigma_V(3,1) = \tau_V(1,3) = \tau_V(3,1) = 0.4,$$

$$\sigma_V(2,3) = \sigma_V(3,2) = 0.3,$$

$$\tau_V(2,3) = \tau_V(3,2) = 0.6,$$

$$\eta_V(1,3) = \eta_V(3,1) = \eta_V(2,3) = \eta_V(3,2) = 0.1.$$

Checking the transitivity,

$$\begin{aligned} \sigma_{U\cup V}(1,3) &\geq \wedge\{\sigma_U(1,2), \sigma_U(2,3)\} \\ &\geq \wedge(0.8, 0.7) \\ &\geq 0.7. \end{aligned}$$

But $\sigma_{U\cup V}(1,3) = 0.6$. Then $\sigma_{U\cup V}(1,3) = 0.6 \ngeq 0.7$. Thus transitivity property fails. So the union of two PFEMRs is not a PFEMR.

4. Conclusions

In this paper, the notion of picture fuzzy equivalence multirelations over picture fuzzy multisets via (r, s, t)-cut set of picture fuzzy multirelations was introduced, and some of the properties related to them were investigated. We have established that a picture fuzzy multirelation over a picture fuzzy multiset is a picture fuzzy equivalence multirelation if and only if the cut set of the picture fuzzy multirelation is an equivalence relation. It was also proved that the two picture fuzzy equivalence classes of a picture fuzzy equivalence multirelation are equal with respect to the cut set of picture fuzzy equivalence multirelation. Furthermore, we have established that the intersection of two picture fuzzy equivalence multirelations on a picture fuzzy multiset is again a picture fuzzy equivalence multirelation on the picture fuzzy multiset but their union need not be a picture fuzzy equivalence multirelation. In future work, applications of PFEMR in decision-making, clustering as well as data analysis will be explored.

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